SOME ASPECTS OF ESTIMATORS IN ANALYSIS OF VARIANCE MODEL II

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THESIS

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ABSTRACT

It is well known that the standard estimator for variance components in analysis of variance, Model II, $\hat{\sigma}_a^2$, can be negative with positive probability. In practice, when such an estimator is found to be negative it is taken to be zero. Very little is known about the properties of the corresponding truncated estimator. This thesis investigates the variance and bias of the positive truncated estimator $\tilde{\sigma}_a^2$. A method of selecting ℓ, the number of classes, is presented that produces maximum power for a test of the hypothesis that $\sigma_a^2 = 0$ while keeping the variance and bias as small as possible.

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I. INTRODUCTION

The use of Model II or the Random Effects Model in analysis of variance can best be described by a simple example. Suppose we draw a sample of ℓ pieces of steel from the population of pieces which have been subjected to a particular annealing process. These ℓ pieces may be considered as a random sample from the population composed of all such pieces of steel which have been or will be produced by this specific process. We might wish to determine the variation of flexural rigidity after the annealing process between the various members of the whole population. If exact measurements of flexural rigidity could be taken from the pieces on hand, the variance could be derived from straight forward statistical methods. However, the experimental methods used to measure flexural rigidity are subject to error. This error is reflected in the fact that if several measurements are taken from one piece of steel, the results are not always exactly the same. In fact, it may be the case that the measurement (experimental) errors are of the same or greater magnitude than the variation we wish to measure between the true rigidities of the different pieces. Using analysis of variance - Model II, it is possible to separate and isolate these two different causes of variation and to obtain an estimate of the true variation of rigidity.

The data for such an analysis will consist of several different measurements of flexural rigidity taken from each

of the ℓ pieces of steel. If we take r measurements on each of the ℓ pieces, then the total number of measurements will be $\ell r = N$.

The Model II analysis of variance now takes the form

$$y_{ij} = \mu + a_i + e_{ij}, \quad i=1,2,\cdots\ell, \; j=1,2,\cdots r. \tag{1.1}$$

The following assumptions and definitions are standard for this model:

Y_{ij} represents the j^{th} measurement of the i^{th} piece of steel,

μ is the "true" mean flexural rigidity of the population and is assumed constant,

a_i is the deviation from the mean of the i^{th} piece. The a_i are assumed to be distributed N $(0, \sigma_a^2)$,

e_{ij} is the measurement error of the j^{th} measurement on the i^{th} piece. The e_{ij} are assumed to be distributed N $(0, \sigma_e^2)$,

a_i and e_{ij} are assumed independent.

For the balanced one-way classification Model II analysis of variance just described, it is well known that the minimum variance unbiased estimator for the true variation between the pieces is

$$\hat{\sigma}_a^2 = \frac{MS_a - MS_e}{r} \tag{1.2}$$

where $MS_a = \sum_{i=1}^{\ell} r(\bar{y}_i - \bar{y})^2/(\ell-1)$

and $MS_e = \sum_{i=1}^{\ell} \sum_{j=1}^{r} (y_{ij} - \bar{y}_i)/\ell(r-1)$.

Leone and Nelson [Ref. 3] found from an empirical study that this estimator can be negative with probability as high

as .40. In practical applications the estimator is taken as zero whenever it is negative. This then produces a truncation of the true distribution of the minimum variance unbiased estimator. The truncated estimator takes the form

$$\tilde{\sigma}_a^2 = \begin{cases} \hat{\sigma}_a^2 & \text{if } MS_a \geq MS_e \\ 0 & \text{otherwise.} \end{cases} \tag{1.3}$$

The properties of this truncated estimator are unknown at present.

Consider a situation where N, the total number of experiments, is fixed. Within this framework, this paper is concerned with an empirical investigation of the properties of the estimator $\tilde{\sigma}_a^2$.

The following questions are considered:

1. What can be said concerning the effects of various choices of ℓ and r on the bias and variance of $\tilde{\sigma}_a^2$?
2. How does the variance of $\tilde{\sigma}_a^2$ compare with the variance of $\hat{\sigma}_a^2$ for a given N and ℓ?
3. Can an allocation method for ℓ be found to yield minimum variance or minimum bias for $\tilde{\sigma}_a^2$?
4. If such an allocation method can be developed, how does it compare with the allocation formula for r developed by Hammersley [Ref. 1] to minimize the variance of $\hat{\sigma}_a^2$ when $K = \dfrac{\sigma_a^2}{\sigma_e^2}$ is known?
5. If nothing is known about K is there a "best" allocation method for ℓ?

6. If we are testing the null hypothesis H_o: $\sigma_a^2 = 0$ against H_1: $\sigma_a^2 \neq 0$, how does the allocation of ℓ affect the power of this test?

II. METHODOLOGY

A. THEORY

In order to investigate the behavior of the estimator it is first necessary to develop the distribution of $\tilde{\sigma}_a^2$ and expressions for the bias and variance of this estimator.

1. Distribution of $\tilde{\sigma}_a^2$

Let U and V be two independent random variables with Pearson Type III distribution so that

$$f(u) = \begin{cases} \dfrac{\gamma_1^{\tau_1+1}}{\Gamma(\tau_1+1)} u^{\tau_1} e^{-\gamma_1 u} & \text{if } u > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{and } f(v) = \begin{cases} \dfrac{\gamma_2^{\tau_2+1}}{\Gamma(\tau_2+1)} v^{\tau_2 \gamma} e^{-\gamma_2 v} & \text{if } v > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Pearson [Ref. 4] found the distribution of Y = U - V to be

$$g(y) = \frac{\gamma_1^{\tau_1+1} \gamma_2^{\tau_2+1}}{\Gamma(\tau_1+1)(\gamma_1+\gamma_2)^{\tau_1+1}} e^{\gamma_2 y} y^{\tau_2} [1 + \frac{\tau_2(\tau_1+1)}{1!(\gamma_1+\gamma_2)y}$$

$$+ \frac{\tau_2(\tau_2-1)(\tau_1+1)(\tau_1+2)}{2!(\gamma_1+\gamma_2)^2 y^2} + \cdots] \quad \text{if } y > 0 \tag{2.1a}$$

$$g(y) = \frac{\gamma_1^{\tau_1+1}\gamma_2^{\tau_2+1}}{\Gamma(\tau_2+1)(\gamma_1+\gamma_2)^{\tau_2+1}} e^{\gamma_1 y}(-y)^{\tau_1}[1 + \frac{\tau_1(\tau_2+1)}{1!(\gamma_1+\gamma_2)(-y)}$$

$$+ \frac{\tau_1(\tau_1-1)(\tau_2+1)(\tau_2+2)}{2!(\gamma_1+\gamma_2)^2(-y)^2} + \cdots] \quad \text{if } y < 0 \; . \tag{2.1b}$$

He also showed that

$$\int_0^\infty g(y)\,dy = I_{\frac{\gamma_1}{\gamma_1+\gamma_2}}(\tau_1+1,\ \tau_2+1) \; , \tag{2.2}$$

$$\int_0^\infty yg(y)\,dy = \frac{\tau_2+1}{\gamma_2} I_{\frac{\gamma_1}{\gamma_1+\gamma_2}}(\tau_1+1,\ \tau_2+1)$$

$$- \frac{\tau_1+1}{\gamma_1} I_{\frac{\gamma_1}{\gamma_1+\gamma_2}}(\tau_1+2,\ \Gamma_2) \tag{2.3}$$

and $$\int_0^\infty e^{ty}g(y)\,dy = (1-\frac{t}{\gamma_2})^{-(\tau_2+1)} (1+\frac{t}{\gamma_2})^{-(\tau_1+1)}$$

$$I_{\frac{\gamma_1+t}{\gamma_1+\gamma_2}}(\tau_1+1,\ \tau_2+1) \; , \tag{2.4}$$

where $I_x(p,q)$ is the ratio of the incomplete beta function to the complete beta function.

If we choose $\gamma_1 = \frac{1}{2b}$, $\tau_1 = \frac{m}{2} - 1$, $\gamma_2 = \frac{1}{2a}$ (2.5)

and $\tau_2 = \frac{n}{2} - 1$,

and substitute these values into 2.1, we obtain the density function of $Y = aX_1 - bX_2$, where X_1 and X_2 are independent chi-square variables with n and m degrees of freedom respectively and a and b are positive constants. This density function becomes

$$g(y) = \begin{cases} \dfrac{1}{\Gamma(\frac{n}{2})\, 2^{\frac{n}{2}} a^{(\frac{n-m}{2})} (a+b)^{\frac{m}{2}}} e^{-y/2a} y^{\frac{n}{2}-1} {}_2F_0[\frac{m}{2}, (1-\frac{n}{2}); \dfrac{-2ab}{(a+b)y}] , & y > 0 \\ \dfrac{1}{\Gamma(\frac{m}{2})\, 2^{\frac{m}{2}} b^{(\frac{m-n}{2})} (a+b)^{\frac{n}{2}}} e^{-y/2a} (-y)^{\frac{n}{2}-1} {}_2F_0[\frac{n}{2}, (1-\frac{m}{2}); \dfrac{2ab}{(a+b)y}] , & y < 0 \end{cases} \tag{2.6}$$

where

$$2F_o[p,q;X] = \sum_{n=0}^{\infty} (p)_n (q)_n \frac{X^n}{n!}$$

with

$$(a)_n = a(a+1) \cdots (a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)} .$$

Now let

$$Y+ = \begin{cases} Y & \text{if } Y \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

The distribution for Y^+ will be

$$H_{Y^+}(y) = \begin{cases} 0 & , \quad y < 0 \\ \int_{-\infty}^{0} g(y)\,dy & , \quad y = 0 \\ \int_{-\infty}^{Y^+} g(y)\,dy & , \quad y > 0 , \end{cases} \tag{2.7}$$

and using Eq. (2.6) we get the density function

$$h_{Y^+}(y) = \begin{cases} I_{\frac{b}{a+b}}(\frac{n}{2}, \frac{m}{2}) & , \quad y = 0 \\ g(y) & , \quad y \quad 0 \\ 0 & , \quad \text{otherwise.} \end{cases} \tag{2.8}$$

From equations (2.3), (2.4), (2.5), and (2.8) it follows that

$$E(Y^+) = na\, I_{\frac{a}{a+b}}(\frac{m}{2}, \frac{n}{2}) - mb\, I_{\frac{a}{a+b}}(\frac{m}{2} + 1, \frac{n}{2} - 1), \tag{2.9}$$

and

$$E(e^{tY+}) = I_{\frac{b}{a+b}}(\frac{n}{2}, \frac{m}{2}) + (1-2at)^{-\frac{n}{2}}(1+2bt)^{-\frac{m}{2}} I_{\frac{a(1+2bt)}{a+b}}(\frac{m}{2}, \frac{n}{2}). \tag{2.10}$$

The distribution of $\tilde{\sigma}_a^2$ is the same as that of Y^+ with

$$n = \ell-1 \qquad m = \ell(r-1),$$

$$a = \frac{\sigma_e^2 + r\sigma_a^2}{r(\ell-1)} \quad \text{and} \quad b = \frac{\sigma_e^2}{r\ell(r-1)} \quad . \tag{2.11}$$

2. Variance and Bias of $\tilde{\sigma}_a^2$.

As indicated above, the distribution of $\tilde{\sigma}_a^2$ is the same as the distribution of Y^+ for the proper choice of n, m, a, and b. Thus, Eqs. (2.9) and (2.10) give the expected value of $\tilde{\sigma}_a^2$ and its moment generating function when $\tilde{\sigma}_a^2$ is substituted for Y^+ and m, n, a, and b are defined as in Eq. (2.11).

The variance of $\tilde{\sigma}_a^2$ can now be derived by recalling that

$$\mathrm{Var}\ (\tilde{\sigma}_a^2) = \left[\frac{d^2M}{dt^2} - \left(\frac{dM}{dt}\right)^2\right]_{t=0} , \qquad (2.12)$$

where $M = E(e^{t\tilde{\sigma}_a^2})$.

Applying successive derivitives to Eq. (2.10) and evaluating at $t = 0$, the expected value of $\tilde{\sigma}_a^2$ is found to be

$$E(\tilde{\sigma}_a^2) = \frac{dM}{dt}\Big|_{t=0} = na\ I_{\frac{a}{a+b}}\left(\frac{m}{2}, \frac{n}{2}\right) - mb\ I_{\frac{a}{a+b}}\left(\frac{m}{2}, \frac{n}{2}\right)$$

$$+ \frac{2ab}{a+b}\left(\frac{a}{a+b}\right)^{\frac{m}{2}-1}\left(\frac{b}{a+b}\right)^{\frac{n}{2}-1}/\beta\left(\frac{m}{2}, \frac{n}{2}\right) \qquad (2.13)$$

where $\beta(\frac{m}{2}, \frac{n}{2})$ is the beta function with parameters m/2 and n/2, and

$$\frac{d^2M}{dt^2}\Big|_{t=0} = (a^2n^2 + 2a^2n - 2abmn + 2b^2m + b^2m^2)\ I_{\frac{a}{a+b}}\left(\frac{m}{2}, \frac{n}{2}\right)$$

$$+ \frac{2ab}{a+b}(an - bm + 2a - 2b)\left(\frac{a}{a+b}\right)^{\frac{m}{2}-1}\left(\frac{b}{a+b}\right)^{\frac{n}{2}-1}/\beta\left(\frac{m}{2}, \frac{n}{2}\right) . \qquad (2.14)$$

Equation (2.13) can be shown to be equivalent to Eq. (2.9). Squaring Eq. (2.13) and subtracting from Eq. (2.14), the variance of $\tilde{\sigma}_a^2$ is obtained as

$$\mathrm{Var}\ (\tilde{\sigma}_a^2) = 2(na^2 + mb^2)\ I_{\frac{a}{a+b}}\left(\frac{m}{2}, \frac{n}{2}\right)$$

$$+ (na - mb)\left[na\ I_{\frac{a}{a+b}}\left(\frac{m}{2}, \frac{n}{2}\right) - mb\ I_{\frac{a}{a+b}}\left(\frac{m}{2} + 1, \frac{n}{2} - 1\right)\right]$$

$$+ 4\left(\frac{a}{a+b}\right)^{\frac{m}{2}}\left(\frac{b}{a+b}\right)^{\frac{n}{2}-1} b(a-b)/\beta\left(\frac{m}{2},\frac{n}{2}\right) - \left[na\ I_{\frac{a}{a+b}}\left(\frac{m}{2},\frac{n}{2}\right)\right.$$

$$\left. - mb\ I_{\frac{a}{a+b}}\left(\frac{m}{2} + 1, \frac{n}{2} - 1\right)\right]^2 . \qquad (2.15)$$

From Eq. (2.9), the bias of the estimator is given by

$$\text{bias} = na\, I_{\frac{a}{a+b}}\left(\frac{m}{2}, \frac{n}{2}\right) - mb\, I_{\frac{a}{a+b}}\left(\frac{m}{2}+1, \frac{n}{2}-1\right) - \sigma_a^2 . \tag{2.16}$$

The expression for the variance of $\hat{\sigma}_a^2$ is

$$\text{Var}\,(\hat{\sigma}_a^2) = \frac{2}{r^2}\left[\frac{(\sigma_e^2 + r\sigma_a^2)^2}{r^2(\ell-1)} + \frac{\sigma_e^4}{\ell(r-1)}\right] . \tag{2.17}$$

Thus, values for the variance of the minimum variance unbiased estimator, $\hat{\sigma}^2$, can be computed for a comparison with the variance of $\tilde{\sigma}_a^2$ for fixed N and ℓ .

3. Power

In considering the problem of selecting an ℓ for a fixed N when testing a given hypothesis based on the sample, the power of the test is an important consideration. Suppose the null hypothesis Ho: $\sigma_a^2 = 0$ is being tested against the alternative hypothesis H_1: $\sigma_a^2 \neq 0$ from a sample of ℓ classes, each class consisting of r observations. From the analysis of variance table, the test statistic is found to be $\mathcal{F} = \frac{MS_a}{MS_e}$, where $MS_a = \frac{SS_a}{\ell-1}$ and $MS_e = \frac{SS_e}{\ell(r-1)}$.

It may be shown that $\frac{SS_a}{\sigma_e^2 + r\sigma_a^2}$ and $\frac{SS_e}{\sigma_e^2}$ are independent chi-square variables with $\ell - 1$ and $\ell(r-1)$ degrees of freedom respectively. The statistic $\frac{MS_a}{MS_e}$ may now be re-written as

$$\mathcal{F} = \frac{\left(\frac{SS_a}{\sigma_e^2 + r\sigma_a^2}\right)/(\ell-1)}{\left(\frac{SS_e}{\sigma_e^2}\right)/\ell(r-1)} . \tag{2.18}$$

and is a ratio of independent chi-square variables divided by their degrees of freedom and is distributed as $F_{(\ell-1),\ \ell(r-1)}$, where $F_{a,b}$ is a central F variable with a and b degrees of freedom.

If H_o is true, i.e., $\sigma_a^2 = 0$ the test statistic

$\mathcal{F} = \frac{SS_a/(\ell-1)}{SS_e/\ell(r-1)}$ is distributed as $F_{(\ell-1),\ \ell(r-1)}$.

Thus, a test of the null hypothesis consists of rejecting H_o at a level of significance α , if

$\mathcal{F} \geq F_{\alpha;(\ell-1),\ \ell(r-1)}$.

The power of this test, denoted $\beta(\theta)$, is given by

$\beta(\theta) = P[\frac{MS_a}{MS_e} > F_{\alpha;(\ell-1),\ \ell(r-1)}]$ where $\theta = \frac{\sigma_a^2}{\sigma_e^2}$.

But if $\sigma_a^2 \neq 0$, then

$$\beta(\theta) = P\left[\frac{\frac{MS_a}{\sigma_e^2 + r\sigma_a^2}}{\frac{MS_e}{\sigma_a^2}} > \frac{\sigma_e^2 F_{\alpha;\ (\ell-1),\ \ell(r-1)}}{\sigma_e^2 + r\sigma_a^2}\right]$$

$$= P\left[F_{(\ell-1),\ \ell(r-1)} > \frac{\sigma_e^2 F_{\alpha;\ (\ell-1),\ \ell(r-1)}}{\sigma_e^2 + r\sigma_a^2}\right]$$

This expression can be evaluated after the transformation

$$Y = \frac{1}{1 + \frac{\gamma_1}{\gamma_2} F_{\gamma 1, \gamma 2}} .$$

The variable Y is distributed as $\beta(\gamma_1, \gamma_2)$.

Thus

$$\beta(\theta) = P\left[\frac{1}{1 + \frac{\ell - 1}{\ell(r-1)} F_{\ell-1, \ell(r-1)}} < \frac{1}{1 + \frac{(\ell-1)\sigma_e^2 F_{\alpha;\ell-1,\ell(r-1)}}{\ell(r-1)(\sigma_e^2 + r\sigma_a^2)}}\right]$$

or

$$\beta(\theta) = I_X[\ell-1, \ell(r-1)]$$

where

$$X = \frac{1}{1 + \frac{(\ell-1)\sigma_e^2 F_{\alpha;(\ell-1),\ell(r-1)}}{\ell(r-1)(\sigma_e^2 + r\sigma_a^2)}}$$

yields the power of the test of hypothesis H_o: $\sigma_a^2 = 0$, for a specified α and N.

B. DATA GENERATION

From Eqs. (2.9), (2.15), (2.16), and (2.19) it can be seen that each of the properties to be analyzed is dependent on four variables: σ_e^2, σ_a^2, r and ℓ. Recall that $N = \ell r$ is the total number of experiments to be conducted. If N is fixed the choice of either ℓ or r determines the other. Thus, for fixed N, we have only three variables, ℓ, σ_e^2 and σ_a^2 . We now wish to see what happens to the bias and variance of $\tilde{\sigma}_a^2$ and the power of the specified test of hypothesis as the three variables take on a range of values.

Calculation of these statistics was done on an IBM 360/67 computer using the basic program shown in Appendix A. In addition to this basic program, the IBM supplied subroutines for computation of the beta-distribution were also used.

The values chosen for N were 12, 16, 20, 40, 50, 80 and 100. For each value of N, ℓ varied through all integral divisors of N such that $4 \leq \ell \leq \frac{N}{2}$. For example, for N = 80, ℓ took on the values 4, 8, 16, 20, and 40.

Initially, for each combination of N and ℓ, both σ_e^2 and σ_a^2 were varied from .1 to 2.0 in steps .1 and again from 1.0 to 20.0 in steps of 1.0. Values were computed for the variance and bias of $\tilde{\sigma}_a^2$, the variance of $\hat{\sigma}_a^2$ and the power of the specified test of hypothesis for all possible combinations of N, ℓ, σ_e^2, and σ_a^2 in the ranges described.

The data generated in this manner supported the contention of Scheffe's that σ_e^2 is simply a scaling factor for both the bias and variance of $\tilde{\sigma}_a^2$ and the variance of $\hat{\sigma}_a^2$. Figures 1 and 2 illustrate the scaling influence of σ_e^2 on the bias and variance of $\tilde{\sigma}_a^2$ when N = 20 and ℓ = 4 and 5.

As for the power of the test of hypothesis, Scheffe' has shown $\beta(\theta)$ to be dependent only upon the ratio σ_a^2/σ_e^2 and again σ_e^2 can be considered as a scaling factor.

Based on these considerations, σ_e^2 was set at one for all data generated for use in this thesis. This greatly reduced the amount of time and output required for computer runs and further reduced the number of input variables to two, ℓ and σ_a^2 for each fixed N. Further, if $\sigma_e^2 = 1$, the value of σ_a^2 is

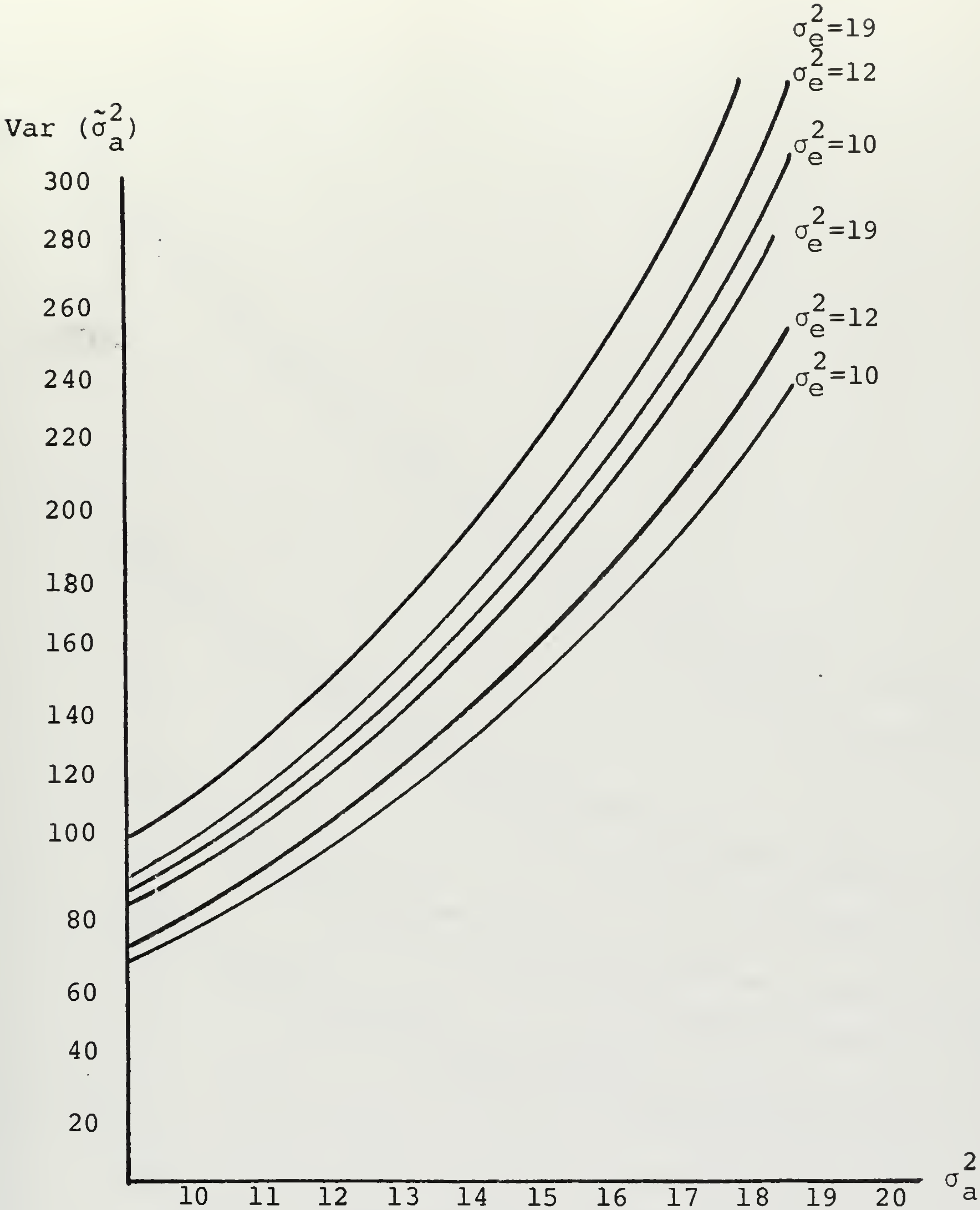

Figure 1. Graph showing the effects of the scaling influence of σ_e^2 on the variance of $\tilde{\sigma}_a^2$ for N = 20. The three upper curves are for ℓ = 4, and the three lower for ℓ = 5.

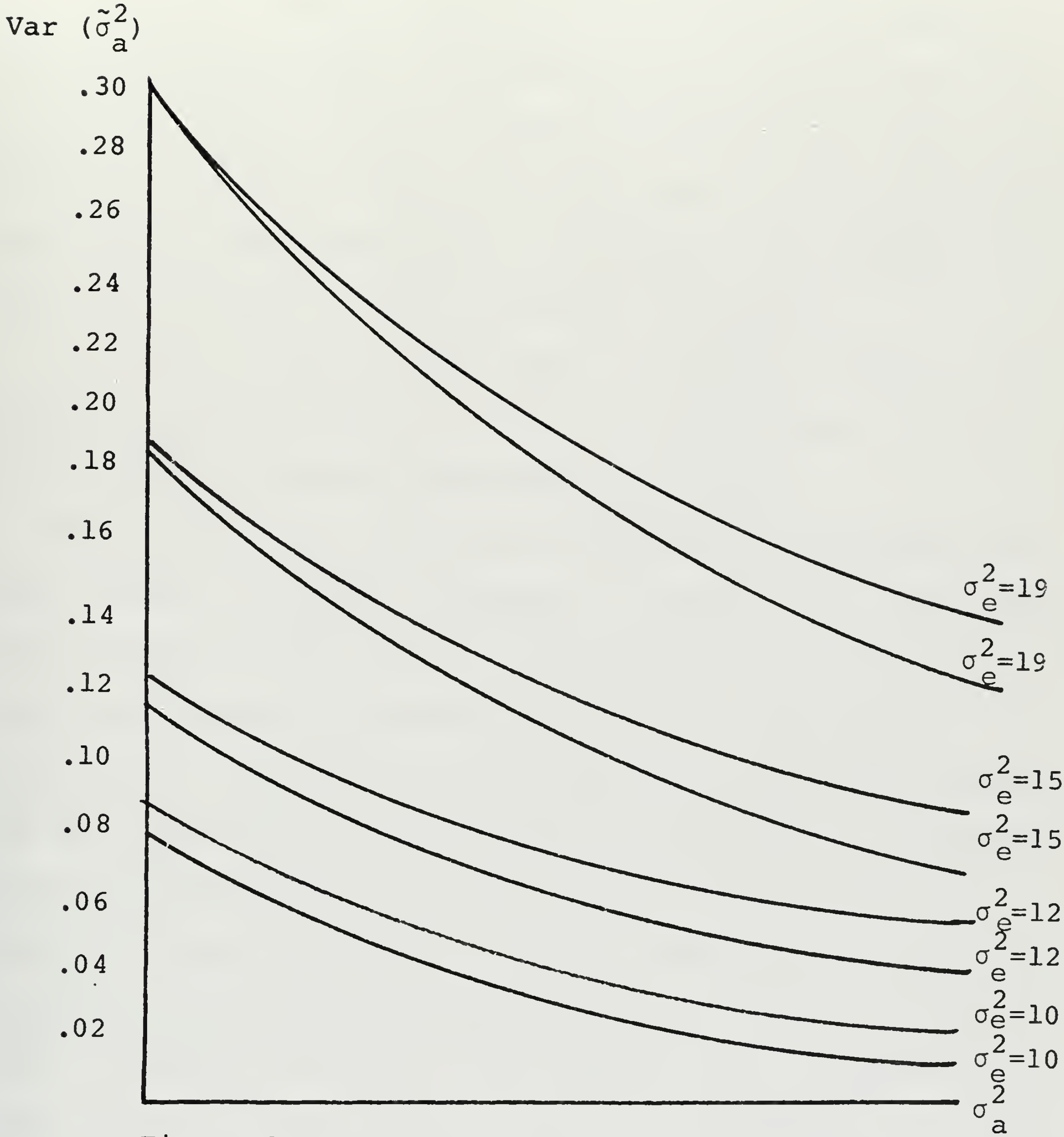

Figure 2. Graph showing the scaling effects of σ_a^2 on the bias of σ_a^2 for N = 20. The top curve in each pair is for ℓ = 4, and the bottom curve for ℓ = 5.

also the value of the ratio of $K = \frac{\sigma_a^2}{\sigma_e^2}$. Since σ_e^2 is a scaling factor, all conclusions drawn using $\sigma_e^2 = 1$ are equally valid for any other σ_e^2 . The direction of change for fixed σ_a^2 is as follows: as σ_e^2 increases (decreases) the variance and bias increase (decrease) and the power of the test of the test of hypothesis decreases (increases). The magnitude of the change depends on the magnitudes of σ_e^2 and σ_a^2 , N and ℓ.

In order to evaluate the power of the test of hypothesis, it was necessary to choose a level of significance, alpha. An alpha of .05 was used throughout this paper.

Wang [Ref. 8] has conducted a similar study of the bias and variance of several estimators, including $\tilde{\sigma}_a^2$. Her study was restricted to the special case where X_1 and X_2 took on only even degrees of freedom and N took on values of 9, 27, 81 and 225. There are no direct points of comparison between the data she generated and the data in this thesis. However, a very favorable comparison of computed variance and bias exists for value of N and ℓ as nearly matching as is possible. Wang's variance and bias expressions for $N = 81$, $\ell = 9$, and $\sigma_a^2 = 1.0$ yield var $(\tilde{\sigma}_a^2) = .309$ and bias $(\tilde{\sigma}_a^2) = 0$ while the data from this thesis for $N = 80$, $\ell = 10$, and $\sigma_a^2 = 1.0$ yield var $(\tilde{\sigma}_a^2) = .282$ and bias $(\tilde{\sigma}_a^2) = 0$.

III. PRESENTATION OF FINDINGS

A. GENERAL OBSERVATIONS

Before attempting to address the specific question of the selection of ℓ, several general conclusions can be drawn from the data regarding the bias of $\tilde{\sigma}_a^2$, the variance of $\tilde{\sigma}_a^2$ and $\hat{\sigma}_a^2$ and the power of the specified test of hypothesis.

Generally speaking, the bias of $\tilde{\sigma}_a^2$ is very small. As shown in Table I, the bias decreases as K, the ratio of σ_a^2/σ_e^2 increases. For small N and small K, the bias is significant. However, if K is greater than 1.0 and N greater than or equal 20, the magnitude of the bias is so small that it can be neglected. For this range of N and K, the maximum value the bias assumed is less than one percent of the true value of σ_a^2. Thus, $\tilde{\sigma}_a^2$ is virtually unbiased in this range.

The bias of $\tilde{\sigma}_a^2$ was found to be negative for many combinations of N, ℓ, and K. However, for the entire range of input variables for which data was generated, the negative bias was always insignificant to the fourth decimal place.

As shown in Table II, the variances of $\tilde{\sigma}_a^2$ and $\hat{\sigma}_a^2$ are very nearly the same except when K is small. For small values of K there is a significant difference between the two. However, this difference decreases sharply as K increases and is negligible for $K \geq 1.0$. The difference between the variances is further decreased as N increases. Thus the variances of $\tilde{\sigma}_a^2$ and $\hat{\sigma}_a^2$, appear to approach each other asymptotically as N and/or K increase.

TABLE I

EFFECT OF N AND K ON THE BIAS OF $\tilde{\sigma}_a^2$

$\sigma_a^2 = K, \ell = 4, \sigma_e^2 = 1.0$

N	K	BIAS	N	K	BIAS
12	.1	.0951	40	.1	.0153
	.5	.0478		.5	.0037
	1.0	.0270		1.0	.0016
	2.0	.0129		2.0	.0006
16	.1	.0627	80	.1	.0045
	.5	.0265		.5	.0010
	1.0	.0138		1.0	.0005
	2.0	.0061		2.0	.0001
20	.1	.0452	100	.1	.0029
	.5	.0167		.5	.0004
	1.0	.0081		1.0	.0002
	2.0	.0035		2.0	.0001

Table II also indicates that the variance of $\tilde{\sigma}_a^2$ is always less than or equal the variance of $\hat{\sigma}_a^2$. The introduction of a small amount of bias by truncation of the estimator tends to reduce the variance.

As is to be expected, the power of the test of hypothesis increases as N increases. In the model proposed here, power is also a function of ℓ when N is fixed. For all values of K tested, it was found that if $N \geq 16$ and $\ell < \frac{N}{2}$, $\beta(\theta) \geq .9996$. This implies that for values of $N \geq 16$ and $\ell < \frac{N}{2}$, the power criterion can be ignored in the selection of ℓ. Attention can then be directed to minimizing variance and/or bias

TABLE II

DIFFERENCES IN TRUNCATED ($\tilde{\sigma}_a^2$) AND UNTRUNCATED ($\hat{\sigma}_a^2$) VARIANCE ESTIMATORS FOR VARIOUS VALUES OF N, K, AND ARBITRARY ℓ

N	K	ℓ	VAR ($\tilde{\sigma}_a^2$)	VAR ($\hat{\sigma}_a^2$)
12	0.1000	4	0.0648	0.1530
12	0.5000	4	0.3461	0.4907
12	1.0000	4	1.0158	1.2130
12	2.0000	4	3.3828	3.6574
12	5.0000	4	18.5601	18.9907
20	0.1000	5	0.0336	0.0696
20	0.5000	5	0.2376	0.2896
20	1.0000	5	0.7305	0.7896
20	2.0000	5	2.4754	2.5396
20	5.0000	5	13.7216	13.7896
40	0.1000	8	0.0170	0.0282
40	0.5000	8	0.1347	0.1425
40	1.0000	8	0.4093	0.4139
40	2.0000	8	1.3831	1.3854
40	5.0000	8	7.7275	7.7282
100	0.1000	20	0.0081	0.0105
100	0.5000	20	0.0525	0.0526
100	1.0000	20	0.1526	0.1526
100	2.0000	20	0.5105	0.5105
100	5.0000	20	2.8473	2.8473

in selecting ℓ with the assurance of a very strong test of the hypothesis.

B. THE SELECTION OF ℓ FOR FIXED N

In the model being studied, it has been assumed that the number of observations of the random variable being observed is fixed at N. Further, it is assumed that r observations will be made on each of ℓ classes of the observable phenomenon so that $\ell r = N$. The problem now arises of how to choose ℓ (or r) so as to obtain the best statistical results. The problem is complicated by the fact that the "best" solution is dependent on the desired result of the analysis. For example, the ℓ that provides the most powerful test of hypothesis for a given N may very well produce maximum bias in our estimate of σ_a^2. In the same manner, the ℓ that provides minimum bias or minimum variance in the estimator may produce a very weak test of the hypothesis that $\sigma_a^2 = 0$.

The selection of ℓ is further complicated by the fact that all of the parameters of interest are dependent on K.

Hammersley [Ref. 1] developed an expression for r which produces minimum variance in $\hat{\sigma}_a^2$, the unbiased estimator of σ_a^2. Equating the first derivative of the expression for the variance of $\hat{\sigma}_a^2$ to zero, Hammersley showed that the integral divisor of N that most nearly satisfies

$$r_h = \frac{(K+1)N + 1}{KN + 2}$$

produces minimum variance in $\hat{\sigma}_a^2$. For the range of N and ℓ used in this study, the value of r_h also produce minimum variance for $\tilde{\sigma}_a^2$.

The r_h proposed by Hammersley has two unpleasant features. First, for some combinations of N and K, the power of the specified test of hypothesis is very low. For example, if K = .5 and N = 12, the power of a test of H_o: $\sigma_a^2 = 0$ is only .2561.

The second and perhaps more serious feature is that Hammersley's solution for r_h requires a knowledge of the ratio σ_a^2/σ_e^2 prior to conducting the intended analysis. In an environment such as the flexural rigidity experiment where the general magnitudes of σ_a^2 and σ_e^2 would be known from previous experiments on similar products this requirement may not be serious. However, for a one-time-only experiment, or an evaluation of a new process this requirement may be completely unreasonable.

The results of the present study indicate that power is maximized for small ℓ while variance is minimized when ℓ assumes its maximum value of N/2. But it has already been shown that power is not a major consideration for $N \geq 16$ if $\ell \neq \frac{N}{2}$. It would appear then that ℓ should be selected very near to but not equal to N/2.

Based on these considerations it appears that

$$\ell_g = [\frac{N}{2} - 1]^-$$

such that $\frac{N}{\ell g}$ is an integer is the "best" choice for ℓ, that is, ℓ_g is the next smaller integral divisor of N. As an example for N = 20

$$\ell_g = [\frac{20}{2} - 1]^- = [9]^- = 5.$$

is the best choice for ℓ. (See Table III.)

TABLE III

VALUES OF ℓ_h AND ℓ_g FOR VARIOUS VALUES OF N AND K

AND THE RESULTING POWER, VARIANCE AND BIAS

GENERATED IN $\tilde{\sigma}_a^2$. σ_a^2 = 1.0 IN ALL CASES

N	K	ℓ(h)	ℓ(g)	Power (h)	Power (g)	Var (h)	Var (g)	Bias (h)	Bias (g)
12	.1	4	4	.9111	.9111	.0648	.0648	.0951	.0951
	.5	6	"	.2561	.9915	.3326	.3461	.0681	.0478
	1.0	6	"	.4971	.9991	.8249	1.0158	.0352	.0270
	2.0	6	"	.7937	1.000	2.4297	3.3827	.0135	.0129
	5.0	6	"	.9804	"	12.0571	18.5601	.0025	.0040
	10.0	6	"	.9987	"	44.0841	70.6058	.0006	.0015
16	.1	4	4	.9996	.9996	.0416	.0416	.0627	.0627
	.5	8	"	.2941	1.000	.2574	.2951	.0462	.0265
	1.0	8	"	.5895	"	.6273	.9273	.0193	.0138
	2.0	8	"	.8841	"	1.7938	3.2109	.0053	.0061
	5.0	8	"	.9958	"	8.6815	18.1115	.0005	.0018
	10.0	8	"	.9999	"	31.5520	69.6657	.0001	.0017
20	.1	5	5	1.000	1.000	.0336	.0336	.0515	.0515
	.5	10	"	.3360	"	.2122	.2376	.0337	.0172
	1.0	10	"	.6725	"	.5074	.7305	.0113	.0073
	2.0	10	"	.9379	"	1.4174	2.4753	.0022	.0025
	5.0	10	"	.9992	"	6.7672	13.7216	.0001	.0005
	10.0	10	"	1.000	"	24.5487	52.4702	.0000	.0001

TABLE III--Continued

N	K	$\ell(h)$	$\ell(g)$	Power (h)	Power (g)	Var (h)	Var (g)	Bias (h)	Bias (g)
40	.1	5	10	1.000	1.000	.0163	.0184	.0220	.0166
	.5	20	"	.5500	"	.1168	.1220	.0089	.0031
	1.0	20	"	.9162	"	.2580	.3482	.0012	.0005
	2.0	20	"	.9983	"	.6825	1.1281	.0000	.0000
	5.0	20	"	1.000	"	3.2092	6.1290	-.0000	.0000
	10.0	20	"	1.000	"	11.6303	33.3543	-.0000	-.0000
80	.1	8	20	1.000	1.000	.0092	.0105	.0052	.0116
	.5	40	"	.8276	"	.0621	.0609	.0012	.0002
	1.0	40	"	.9968	"	.1278	.1665	.0000	.0000
	2.0	40	"	1.0000	"	.3330	.5350	-.0000	-.0000
	5.0	40	"	1.0000	"	1.5638	2.9034	-.0000	-.0000
	10.0	40	"	1.0000	"	5.6663	11.0613	-.0000	-.0000
100	.1	10	25	1.000	1.000	.0096	.0119	.0034	.0085
	.5	50	"	.8991	"	.0501	.0485	.0005	.0001
	1.0	50	"	.9995	"	.1018	.1319	.0000	-.0000
	2.0	50	"	1.0000	"	.2651	.4235	-.0000	"
	5.0	50	"	1.0000	"	1.2447	2.2985	-.0000	"
	10.0	50	"	1.0000	"	4.5100	8.7569	-.0000	"

Table III shows a comparison of $\ell_h (\ell_h = \frac{N}{r_h})$ and ℓ_g for various values of N. It also shows the power, variance, and bias generated for each choice of ℓ_g and ℓ_h in a range of K values. It can be seen that ℓ_h increases with K to its maximum value of N/2 while ℓ_g is fixed for a given N. Also, when ℓ_h = N/2, the power of the test is small for small N. In fact, for N as large as 100, the power using ℓ_h may be less than .9 while power for ℓ_g never falls below .9 for any N. As was expected, the variance of $\tilde{\sigma}_a^2$ using ℓ_h is considerably less than the variance acquired using ℓ_g since ℓ_h was derived as the minimum variance choice of ℓ.

Generally speaking, the bias of the estimator when $\ell = \ell_g$ is less than or equal the bias when $\ell = \ell_h$. The only exceptions to this being when K = .1 and N = 80 and 100. The bias for both $\ell = \ell_g$ and $\ell = \ell_h$ is generally less than three percent of the true value of σ_a^2 if $K \geq 1.0$ and less than one percent for $K \geq 1.0$ and $N \geq 20$.

Again it seems that the method of selecting ℓ depends on the desired results of the original analysis. ℓ_h will always produce minimum variance in the estimate of σ_a^2 but requires a knowledge of the ratio $K = \sigma_a^2/\sigma_e^2$. If K is known and a minimum variance estimator is desired, this is certainly the best method of choosing ℓ.

If a powerful test of hypothesis is desired ℓ_g gives a much more powerful test for most combinations of N and K than will ℓ_h . If nothing is known of K, ℓ_g gives a powerful test and a relatively small variance.

IV. CONCLUSIONS AND RECOMMENDATIONS

A. CONCLUSIONS

It may be concluded that with one exception, ℓ_g is the "best" method of choosing the number of classes for Model II analysis of variance when N is fixed. The exception occurs in the case where K is known and a minimum variance estimator of σ_a^2 is desired without regard to the power of the test of hypothesis that $\sigma_a^2 = 0$. In this case ℓ_h appears best. The use of $\ell = \ell_g$ assures a very powerful test of hypothesis and will yield a small, but not minimum variance in the estimator. For most combinations of N and K, ℓ_g also produces minimum bias in $\tilde{\sigma}_a^2$.

If $N \geq 20$ and $K \geq 1.0$, the bias of $\tilde{\sigma}_a^2$ is so small as to be negligible. In such cases, the use of the truncated estimator of σ_a^2 has no significant influence on the results of the analysis except to cause a small decrease in variance. As N and/or K increase, this decrease in variance appears to tend toward zero.

B. RECOMMENDATIONS FOR FURTHER STUDY

It is suggested that a similar study of variance estimators be conducted for value of N greater than 100 for the full range of K values studied here. Such a study might also investigate values of K less than the minimum value of .1 used in this study.

A much more difficult task that could follow the same general approach would be an investigation of two-way and multi-way analysis in an effort to determine the best number of experiments for each class to provide minimum variance in the variance estimators and maximum power for a specified test of hypothesis.

APPENDIX A

THIS PROGRAM IS DESIGNED TO COMPUTE THE VARIANCE, POWER AND BIAS FOR THE Y+ ESTIMATOR OF THE BETWEEN CLASS VARIANCE FOR THE BALANCED, ONE-WAY ANALYSIS OF VARIANCE, MODEL II.

EXPLANATION OF SYMBOLS:

VAR IS THE TRUE WITHIN CLASS VARIANCE.

VARA IS THE TRUE BETWEEN CLASS VARIANCE.

L IS THE NUMBER OF CLASSES.

IR IS THE NUMBER OF EXPERIMENTS IN EACH CLASS.

XK IS THE RATIO OF THE BETWEEN CLASS AND WITHIN CLASS VARIANCES.

N IS THE TOTAL NUMBER OF EXPERIMENTS. N=L*IR.

VART IS THE Y+, OR POSITIVE TRUNCATION, OF THE ESTIMATE OF THE BETWEEN CLASS VARIANCE.

VARR IS THE MINIMUM VARIANCE UNBIASED ESTIMATOR OF THE BETWEEN CLASS VARIANCE.

XMEAN IS THE EXPECTED VALUE OF Y+.

POW IS THE POWER OF THE TEST OF HYPOTHESIS THAT THE TRUE BETWEEN CLASS VARIANCE IS ZERO.

XC IS THE F-STATISTIC FOR ALPHA=.05 AND L-1 AND L*(IR-1) DEGREES OF FREEDOM, USED IN COMPUTING THE POWER.

THE SUBROUTINE BDTR COMPUTES THE PROBABILITY THAT THE RANDOM VARIABLE U, DISTRIBUTED ACCORDING TO THE BETA-DISTRIBUTION WITH PARAMETERS A AND B, IS LESS THAN OR EQUAL TO X; BDTR(X,A,B)

THE FUNCTION EYPLUS COMPUTES THE EXPECTED VALUE OF Y+.

THE FUNCTION VYPLUS COMPUTES THE VARIANCE OF Y+.

```
1002 READ(5,105)L,IR,XC
     VAR=1.
     VARA=0.
     IF(L)1000,1000,1001
1001 WRITE(6,100)
     XN=L-1
     XM=L*(IR-1)
     DO 3000 I=1,20
     VARA=FLOAT(I)
     A=(VAR+FLOAT(IR)*VARA)/FLOAT(IR*(L-1))
     B=VAR/FLOAT(L*IR*(IR-1))
     XMEAN=EYPLUS(XM,XN,A,B)
     BIAS=XMEAN-VARA
     VART=VYPLUS(XM,XN,A,B,XMEAN)
     VARR=(2./FLOAT(IR**2))*((VAR+FLOAT(IR)*VARA)**2/
     1FLOAT(L-1)+VAR**2/FLOAT(L*(IR-1)))
     X=1./(1+XN*XC*VAR/(XM*(VAR+FLOAT(IR)*VARA)))
     CALL BDTR(X,XN,XM,P,D,IER)
     POW=P
     N=L*IR
     XK=VARA/VAR
     WRITE(6,101)N,L,XK,VARA,POW,VARR,VART,BIAS
     GO TO 1002
1000 CONTINUE
 100 FORMAT('  N    L    XK    VARA    POWER    VAR(UNTRUN)
    1   BIAS(TRUN)'//' ')
 101 FORMAT(' ',I3,2X,I2,3F8,4,2X,F8.4,2X,F8.4,2X,F8.4)
 105 FORMAT(I2,I4,F6.2)
     END


     FUNCTION EYPLUS(XM,XN,A,B)
     C=A/(A+B)
     H=XM/2.
     E=XN/2.
     CALL BDTR(C,H,E,P,D,IER)
     BET=P
     F=H+1.
     G=E-1.
     IF(G)5,5,10
   5 EYPLUS=0.
     RETURN
  10 CALL BDTR(C,F,G,P,D,IER)
     BET1=P
     EYPLUS=XN*A*BET-XM*B*BET1
     RETURN
     END
```

```
      FUNCTION VYPLUS(XM,XN,A,B,XMEAN)
      IF(XMEAN)10,5,10
    5 EYPLUS=0.
      RETURN
   10 C=XM/2.
      H=XN/2.
      E=A/(A+B)
      F=XN*A**2+XM*B**2
      CALL BDTR(E,C,H,P,D,IER)
      BET=P
      G=XN*A-XM*B
      YM=XM/2.
      YN=XN/2.
      GANM=YM+YN
      CALL GMMMA(YM,GX,IER)
      GAM=GX
      CALL GMMMA(YN,GX,IER)
      GAN=GX
      CALL GMMMA(GANM,GX,IER)
      COG=GAM*GAN/GX
      P=E**C
      R=(1.-E)**(H-1.)
      XK=COG*P*R*2.*B
      VYPLUS=2.*F*BET+XMEAN+2.*XK*(A-B)-XMEAN**2
      RETURN
      END
```

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13. ABSTRACT

It is well known that the standard estimator for variance components in analysis of variance, Model II, $\hat{\sigma}_a^2$, can be negative with positive probability. In practice, when such an estimator is found to be negative it is taken to be zero. Very little is known about the properties of the corresponding truncated estimator. This thesis investigates the variance and bias of the positive truncated estimator $\tilde{\sigma}_a^2$. A method of selecting ℓ, the number of classes, is presented that produces maximum power for a test of the hypothesis that $\sigma_a^2 = 0$ while keeping the variance and bias as small as possible.

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